Lefschetz Hyperplane Theorem for Tropical Abelian Varieties

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Introduction: We establish, using the stratified Morse theory of Goresky and MacPherson, a Lefschetz hyperplane theorem for principally polarized tropical abelian varieties. We also explore the existence of Lefschetz-type theorems for more general subsets of tropical abelian varieties.

We are inspired by, and build upon the work of Adiprasito and Björner, who proved a Lefschetz hyperplane theorem for tropical projective varieties.

Lefschetz for Bergman fans

A **Bergman fan** is the tropical analog of a linear space. More precisely, let $\mathcal{M}$ be a rank r matroid with support $|\mathcal{M}| = \{1, ..., m\}$, the rays of the Bergman fan of $\mathcal{M}$ are generated by vectors in $\mathbb{R}^m/\mathbb{R}\mathbb{1}$ of the form

$$e_F = \sum_{i \in F} e_i$$

where F is a flat of $\mathcal{M}$. The cones are spanned by collections of rays associated to proper chains of flats. These cones realize the Bergman fan as a fan of dimension $r-1$ in $\mathbb{R}^m/\mathbb{R}\mathbb{1}$.

As noticed by Adiprasito and Björner, Bergman fans satisfy a Lefschetz hyperplane theorem.

Theorem (A-B, 2015)**.** *Let $\mathcal{F}$ be the Bergman fan of $\mathcal{M}$. Let H^+ be a generic halfspace in $\mathbb{R}^m/\mathbb{R}\mathbb{1}$ with $\vec{0} \in \partial H^+$. Then, $\mathcal{F} \cap H^+$ is homotopy Cohen-Macaulay of dimension $r-2$. In particular, it is homotopy equivalent to a wedge sum of spheres of dimension $(r-2)$.*

PPTAVs and their subvarieties

Definition. ***A principally polarized tropical abelian variety*** *(PPTAV) is a pair $\mathcal{A} = (\mathbb{R}^n/\Lambda, Q)$: Λ is a rank n lattice and Q is a positive-definite quadratic form on $\mathbb{R}^n$.*

Given a pair (Λ, Q) of lattice and positive-definite quadratic form, there is an associated **Voronoi decomposition**: a tilling of $\mathbb{R}^n$ composed of the translates by Λ of the **Voronoi polytope**

$$\mathrm{Vor}(Q) = \{x \in \mathbb{R}^n : Q(x) \leq Q(x-\lambda) \text{ for all } \lambda \in \Lambda\}.$$

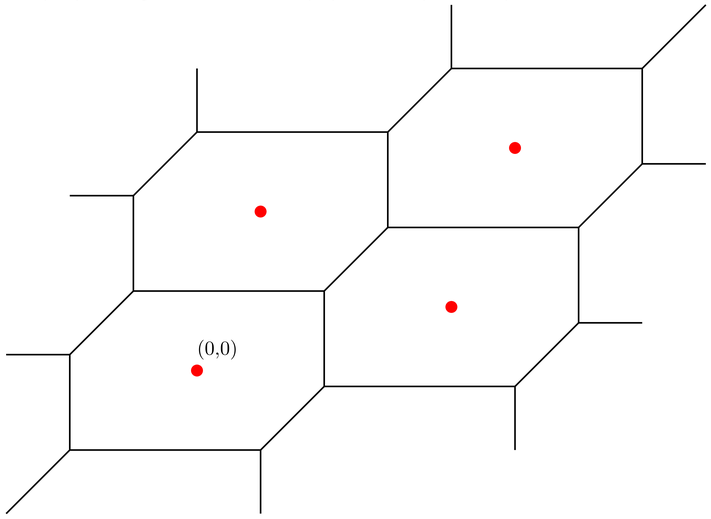

Figure 1: An example of a Voronoi Decomposition

The **theta divisor** of $\mathcal{A} = (\mathbb{R}^n/\Lambda, Q)$, namely Θ, is the (codimension 1)-skeleton of the Voronoi decomposition, mod Λ. Mikhalkin and Zharkov showed that Θ is the bending locus of the tropical theta function.

Let X be a subspace of $\mathbb{R}^n/\Lambda$, and let $\widetilde{X}$ be the lift of X to the universal cover $\mathbb{R}^n$. The subspace X is a **smooth subvariety** of $\mathcal{A}$ if $\widetilde{X}$ is a Λ-periodic polyehesatisfies-dral complex of pure dimension d, such that X is locally modeled on a Bergman fan.

Stratified Morse theory

Let Z be a Whitney stratified subspace of a n-manifold M, and let $\tilde{f} : M \to \mathbb{R}$ be smooth and proper. Roughly speaking, the function $f = \tilde{f}|_Z$ is a **stratified Morse function** on Z if $f|_S$ is a Morse function on each stratum S of Z, and if it satisfies some compatibility condition with respect to the stratification on Z.

We prove a Lefschetz hyperplane theorem for principally polarized tropical abelian varieties using stratified Morse theory, with the theta divisor playing the role of ample divisor. A key argument is the existence of Morse functions on smooth subvarieties that interact nicely with the theta divisor.

Such functions are in fact generic, and we can require them to have maximal Morse index at critical points. Then, by applying Lefschetz for Bergman fans, we obtain the desired result.

Theorem (Lefschetz for PPTAVs)**.** *Let X be a dimension d smooth subvariety of $\mathcal{A} = (\mathbb{R}^n/\Lambda, Q)$. Let Θ be the theta divisor of $\mathcal{A}$. Then, up to homotopy, X can be obtained from $X \cap \Theta$ by attaching cells of dimension d.*

In particular, the inclusion $X \cap \Theta \hookrightarrow X$ induces an isomorphism of homotopy groups of dimensions $\leq d-2$.

Beyond smooth subvarieties

One could ask if it is possible to find more general subsets of tropical abelian varieties satisfying a Lefschetz hyperplane theorem. We construct examples of such subsets, all of which are examples of **d-Lefschetz polyhedral subsets**.

Motivated by tropical Brill-Noether theory, we also propose the following conjecture.

Conjecture. *Let X be the intersection of k translates of Θ. Then up to homotopy, the torus $\mathbb{R}^n/\Lambda$ can be obtained from X by attaching cells of dimensions $\geq n-k$.*